Asymptotic Ergodic Capacity in MIMO Binomial Networks

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Abstract—In this paper, we characterize the asymptotic capacity of a multiple-input multiple-output (MIMO) channel with random locations of nodes in a binomial field of networks. We first derive the high signal-to-noise ratio approximate expression for the MIMO capacity. Then, we characterize the asymptotic capacity of the $n$th nearest receiver as the number of antennas tends to infinity.

I. INTRODUCTION

The random deployment of communicating nodes is one of inevitable uncertainty in wireless networks. The Poisson point process (PPP) has been shown to be a good model for random locations of the nodes in wireless networks [1]–[3]. However, there exist some shortcomings of this Poisson model in practical scenarios, for example, the random set of node locations in a finite region is often non-stationary and the node numbers in disjoint areas are not independent if the number of nodes in the network is known. Since a wireless network is usually formed by a fixed number of nodes in a bounded region—i.e., finite in both the number of users and the serving region (e.g., femtocell networks [4])—the binomial point process (BPP) has been suggested to model these random node locations in a finite wireless network [5]. In this paper, we characterize the asymptotic capacity of a multiple-input multiple-output (MIMO) channel with accounting for random locations of nodes in a binomial field.

II. NETWORK MODEL

We consider a MIMO wireless network, where a transmitter with $n_t$ antennas conveys information messages to $n_r$-antenna receivers randomly located in a two-dimensional compact disk $R$ of radius $R$ centered at the transmitter. Then, the received signal vector at the $n$th nearest node can be written as

$$
y_n = \sqrt{\frac{\gamma}{n_t R^2}} H_n x + z_n$$

where $x$ is the $n_t \times 1$ transmitted signal vector with $E\{||x||^2\} \leq 1$; $H_n \sim \mathcal{N}_{n_t,n_t}(0, I_{n_t}, I_{n_t})$ is the $n_t \times n_t$ Rayleigh-fading channel matrix between the transmitter and the $n$th nearest node; $z_n \sim \mathcal{N}_{n_t,1}(0, I_{n_r}, 1)$ is the zero-mean additive white Gaussian noise vector; $\gamma$ denotes the average received signal-to-noise ratio (SNR) per receive antenna measured at a reference distance of 1 meter away from the transmitter; and $\alpha \geq 2$ is the power loss exponent. In what follows, we refer to $p = \min(n_t, n_r)$ and $q = \max(n_t, n_r)$. We consider that the channel state information is known at the receivers but unknown at the transmitter. We also assume that the fading processes of all links are ergodic and independent.

Let $R_n$ be the distance of the $n$th nearest node in the compact disk $R$. Then, $R_n$ follows the generalized beta distribution [5, Corollary 2.2].

III. MIMO CAPACITY FOR THE BPP

In this section, we analyze the asymptotic ergodic capacity per receive antenna of the $n$th nearest receiver as the antenna numbers tend to infinite in the binomial field MIMO network.

The ergodic capacity in nats per second per hertz (nats/s/Hz) of the $n$th nearest node $(n = 1, 2, \ldots, N)$ in the binomial field MIMO network with the serving region $R$ can be defined as

$$\langle C \rangle_n (R) = E_{R_n, H_n} \left\{ \ln \det \left( I_{n_r} + \frac{\gamma}{n_t R_n^2} H_n H_n^H \right) \right\}.$$  \hspace{1cm} (2)

Theorem 1: In high-SNR regime, the MIMO capacity in (2) can be approximated as

$$\langle C \rangle_n (R) \approx E_{R_n, W_n} \left\{ \ln \det \left( \frac{\gamma}{n_t R_n^2} W_n \right) \right\}.$$  \hspace{1cm} (3)

where $W_n \sim \mathcal{W}_p(q, I_p)$. Using the moments of $\log \det (W_n)$ and $\det (A) = \alpha^k \det (A)$ for $A \in \mathbb{C}^{k \times k}$, we have

$$\langle C \rangle_n (R) \approx p \ln \left( \frac{\gamma}{n_t} \right) + \sum_{i=0}^{p-1} \psi(q-i) - p \zeta(\alpha, n, N, R)$$

where

$$\zeta(\alpha, n, N, R) = \frac{\alpha}{2} (\psi(n) - \psi(N + 1) + 2 \ln R).$$  \hspace{1cm} (5)

Note that $p \zeta(\alpha, n, N, R)$ is the high-SNR slope shift gain (power-offset) which is caused by random locations of nodes in a binomial field of networks.

As the antenna numbers $n_r$ and $n_t$ tend to infinity in such a way that $n_t/n_r \to \tau$, the empirical density for the eigenvalues of $n_t^{-1} H_n H_n^H$ converges almost surely to the Marčenko–Pastur density:

$$f^* (z; \tau) = (1-\tau)^+ \delta(z) + \frac{\sqrt{z-a_\tau}^+ (b_\tau-z)^+}{2\pi z}.$$  \hspace{1cm} (6)

where $a_\tau = (1-\sqrt{\tau})^2$ and $b_\tau = (1+\sqrt{\tau})^2$. 


Fig. 1. Ergodic capacity $\langle C \rangle_n (R)$ in nats/s/Hz of the $n$th nearest node and its scaled asymptote $n_t \langle C \rangle_n^* (R; \tau)$ as a function of $n$ when $R = 10$, $N = 10$, $\gamma = 50 \text{ dB}$, $\alpha = 4$, and $n_t = n_t = 2, 3, 4$ ($\tau = 1$).

\textbf{Theorem 2:} Let

$$\langle C \rangle_n^* (R; \tau) \triangleq \lim_{n_t \to \infty} \frac{\langle C \rangle_n (R)}{n_t} \quad (7)$$

be the asymptotic capacity per receive antenna of the $n$th nearest MIMO link. Then, $\langle C \rangle_n^* (R; \tau)$ in nats/s/Hz per receive antenna for the binomial field MIMO network is given by (8), where the parameter sequences $a_n^*$ and $b_n^*$ are given by

$$a_n^* = (\Delta (\nu, 1 - n), \Delta (2\nu, 0), \Delta (2\nu, 1)) \quad (9)$$

$$b_n^* = (\Delta (2\nu, 0), \Delta (2\nu, 0), \Delta (\nu, -N)) \quad (10)$$

\textbf{IV. NUMERICAL RESULTS AND DISCUSSION}

Fig. 1 shows the ergodic capacity $\langle C \rangle_n (R)$ in nats/s/Hz of the $n$th nearest node and its scaled asymptote $n_t \langle C \rangle_n^* (R; \tau)$ accurately predicts the exact results even for two antennas. In Fig. 2, ergodic capacity $\langle C \rangle_n (R)$ in nats/s/Hz of the nearest node and its high-SNR approximation as a function of $\gamma$ are depicted when $R = 10$, $N = 10$, $\alpha = 4$, and $n_t = n_t = 2, 3, 4$. For comparison, the ergodic capacity of the nearest node in the Poisson field network (corresponding spatial density $\lambda = 0.032$) [6, eq. (13)] is also plotted, which gives a tight lower bound on the ergodic capacity since the serving region in our model is circular. We can observe that the high-SNR approximation in Theorem 1 is remarkably accurate in high-SNR regime.

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